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Apparently first closed-form solution for vibrating: inhomogeneous beams

I. Elishakoff [a,*], S. Candan [b]

[a] *Department of Mechanical Engineering, Florida Atlantic University, 777 Glades Road, P.O. Box 3091, Boca Raton, FL 334312-0991, USA*

[b] *Laboratoire de Recherches et Applications en Mécanique Avancée, Institut Francais de Mécanique Avancée, Aubière F-63175, France*

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Abstract

Free vibration of non-uniform beams, which possess non-homogeneous material density and elastic modulus along their axis, are studied under various boundary conditions. Closed-form expressions for the fundamental natural frequency are derived. It is shown that there is an infinite number of beams that share the same natural frequency. Moreover, it is proved that some coefficients describing the density and elastic modulus functions can be deterministic or random, yet, remarkably, in special circumstances, the fundamental natural frequencies turn out to be deterministic quantities. Extensive numerical analysis is performed to substantiate this seemingly paradoxical finding by the Monte Carlo method, Boobnov–Galerkin method and the finite-element method.

Keywords: Vibrations; Beams; Closed-form solutions

1. Introduction

There are several articles that deal with the vibrations of beams that are non-uniform along their axes. Usually, the variation is attributed to the cross-sectional area. Then, for specific analytic expressions of such deterministic variations, exact solutions are given in terms of special functions. The first solution for the natural frequency of a tapered beam, that of the wedge, was pioneered by Kirchhoff (1882). The solution was given in terms of Bessel functions. Several other solutions of Kirchhoff's type followed. The appropriate bibliography of problems solved in terms of Bessel functions is given by Naguleswaran (1994). The case where hypergeometric functions arise is discussed by Wang (1967). In some special cases, for beams that are clamped at both ends, a transformation of the dependent variable is possible so that the tapered beam shares a natural frequency of the uniform one (Abrate, 1995). There are fewer studies that deal, in various approximate settings, with vibrations of beams with random non-homogeneities. For example, Collins and Thomson (1969), Shinozuka and Astill (1972), and Manohar and Keane (1993)

* Corresponding author. Tel.: +1-561-297-2729; fax: +1-561-297-2825.
E-mail address: ielishak@me.fau.edu (I. Elishakoff).

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considered several random eigenvalue problems via analytical methods. Finite element method in stochastic setting was applied, amongst others, by Hart and Collins (1970), Nakagari et al. (1987), and Zhu and Wu (1991). For extensive bibliography devoted to vibrations of discrete and continuous structures with random parameters one may refer the review by Ibrahim (1987) and its recent update by Manohar and Ibrahim (1997).

Present study deals with beams which exhibit inhomogeneity both in material density and in elastic modulus. These inhomogeneities are described in terms of polynomial functions. Simple closed-form expressions for both mode shapes and fundamental natural frequencies are uncovered for special class of problems. Then the problem is considered in the probabilistic setting with attendant seemingly paradoxical conclusion: beams with random properties may possess the deterministic fundamental natural frequencies. Extensive numerical study is conducted to substantiate this finding.

2. Formulation of the problem

Consider a non-uniform beam of length L, with cross-sectional area A, moment of inertia I, that are constant, and variable material density $\rho(x)$ and modulus of elasticity $E(x)$. Beam's vibrations are governed by the Bernoulli–Euler equation:

$$\frac{\partial^2}{\partial x^2}\left[E(x)I\frac{\partial^2 w(x,t)}{\partial x^2}\right]+\rho(x)A\frac{\partial^2 w(x,t)}{\partial x^2}=0, \tag{1}$$

where $w(x,t)$ is the displacement, x, the axial coordinate, and t, the time.

We introduce non-dimensional coordinate $\xi = x/L$, as well as consider harmonic vibration, so that the displacement $w(\xi,t)$ is represented as follows:

$$w(\xi,t)=W(\xi)\mathrm{e}^{\mathrm{i}\omega t}, \tag{2}$$

where $W(\xi)$ is the mode shape and ω, the sought natural frequency. Thus, Eq. (1) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[E(\xi)\frac{\mathrm{d}^2W(\xi)}{\mathrm{d}\xi^2}\right]-kL^4\rho(\xi)W(\xi)=0, \quad k=\omega^2A/I. \tag{3}$$

The material density and elastic modulus are represented as polynomial functions,

$$\rho(\xi)=\sum_{i=0}^{m}a_i\xi^i, \qquad E(\xi)=\sum_{i=0}^{n}b_i\xi^i, \tag{4}$$

where m and n are positive integers. We restrict our consideration to the case $n = m + 4$, since the first term in Eq. (1) involves four spatial derivatives. We are looking for that special class of problems in which the mode shape $W(\xi)$ is represented by the simplest polynomial functions which satisfy a given set of boundary conditions.

3. Cantilever beam

The beam is clamped at $\xi = 0$ and free at $\xi = 1$. The boundary conditions read

$$\begin{aligned}&W(\xi)=0, \quad \frac{\partial W(\xi)}{\partial \xi}=0 \qquad \text{at } \xi=0,\\ &E(\xi)I\frac{\partial^2W(\xi)}{\partial\xi^2}=0, \quad \frac{\partial}{\partial\xi}\left[E(\xi)I\frac{\partial^2W(\xi)}{\partial\xi^2}\right]=0 \qquad \text{at } \xi=1.\end{aligned} \tag{5}$$

A polynomial function that satisfies the boundary conditions in Eq. (5) is given by

$$W(\xi) = w_1\left(6\xi^2 - 4\xi^3 + \xi^4\right). \tag{6}$$

It coincides with the expression of the first comparison function in the set of polynomial functions introduced by Duncan (1937) for studying beam vibration in the context of the Boobnov–Galerkin method; w_1 is an indeterminate coefficient. We also note that the expression in parentheses is proportional to the static displacement of the uniform cantilever under the constant loading.

We pose the following question: What should the coefficients a_i and b_i be so that the beam's vibration mode coincides with Eq. (6)?

By substituting the expressions for $E(\xi)$, $\rho(\xi)$ and $W(\xi)$ into Eq. (3), we obtain

$$w_1\left[\sum_{i=2}^{m+4} i(i-1)b_i\xi^{i-2}\left(12\xi^2 - 24\xi + 12\right) + \sum_{i=0}^{m+4} 24b_i\xi^i + 2\sum_{i=1}^{m+4} ib_i\xi^{i-1}(24\xi - 24) - kL^4\sum_{i=0}^{m} a_i\xi^i\left(\xi^4 - 4\xi^3 + 6\xi^2\right)\right] = 0. \tag{7}$$

The latter expression can be re-written as follows, in a more convenient form:

$$-24\sum_{i=1}^{m+3} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{m+4} i(i-1)b_i\xi^i + 12\sum_{i=0}^{m+2}(i+2)(i+1)b_{i+2}\xi^i + 24\sum_{i=0}^{m+4} b_i\xi^i - 48\sum_{i=0}^{m+3}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{m+4} ib_i\xi^i - 6kL^4\sum_{i=2}^{m+2} a_{i-2}\xi^i + 4kL^4\sum_{i=3}^{m+3} a_{i-3}\xi^i - kL^4\sum_{i=4}^{m+4} a_{i-4}\xi^i = 0. \tag{8}$$

Eq. (8) has to be satisfied for any ξ. It will be shown later that one has to distinguish two special sub-cases: (a) $m \leqslant 3$ and (b) $m > 3$. It appears instructive to first treat the particular cases $m = 0$, 1, 2 and 3.

3.1. Cantilever with uniform mass density ($m = 0$)

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0, \qquad E(\xi) = \sum_{i=0}^{4} b_i\xi^i. \tag{9}$$

By substituting the latter expressions in Eq. (3), we obtain

$$-24\sum_{i=1}^{3} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{4} i(i-1)b_i\xi^i + 12\sum_{i=0}^{2}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{4} b_i\xi^i - 48\sum_{i=0}^{3}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{4} ib_i\xi^i - kL^4a_0\left(6\xi^2 - 5\xi^3 + 2\xi^4\right) = 0. \tag{10}$$

Eq. (10) has to be satisfied for any ξ. This requirement yields

$$24(b_0 + b_2) - 48b_1 = 0, \tag{11}$$

$$72(b_1 + b_3) - 144b_2 = 0, \tag{12}$$

$$144(b_2 + b_4) - 288b_3 - 6kL^4a_0 = 0, \tag{13}$$

$$240b_3 - 480b_4 + 4L^4ka_0 = 0, \tag{14}$$

$$-kL^4a_0 + 360b_4 = 0. \tag{15}$$

The sole unknown in Eqs. (11)–(15) is the natural frequency coefficient k, yet we have five equations. We conclude that the parameters b_i and a_i have to satisfy some auxiliary conditions so that Eqs. (11)–(15) are compatible.

Two compatibility conditions are given by Eqs. (11) and (12), leading to

$$b_0 = 2b_1 - b_2, \tag{16}$$

$$b_1 = 2b_2 - b_3. \tag{17}$$

From Eqs. (13)–(15), three expressions for k can be found. These are listed below:

$$k = \frac{144(b_2 + b_4) - 288b_3}{6L^4a_0}, \tag{18}$$

$$k = \frac{-240b_3 + 480b_4}{L^44a_0}, \tag{19}$$

$$k = \frac{360}{L^4a_0}b_4. \tag{20}$$

For satisfying the compatibility requirement, all expressions for k have to be equal to each other. We consider the case when material density coefficients a_i are specified. Then problem is reduced to determining coefficients b_i so that Eqs. (18)–(20) are compatible.

Since the function $\rho(\xi)$, of the material density, is given, so is the coefficient a_0. Let us observe that if b_4 is specified then the expression given in Eq. (20) is the final formula for the natural frequency coefficient k. Then, Eqs. (18)–(20) allow an evaluation of remaining parameters b_i. Note that b_4 and a_0 have to have the same sign since the natural frequency parameter k must be positive. From Eq. (19), we obtain

$$b_3 = -4b_4. \tag{21}$$

Eq. (18) leads to

$$b_2 = -\frac{b_2}{2} + 4b_4. \tag{22}$$

The b_is, where $i = \{0, 1, 2, 3\}$, can be re-written as follows:

$$b_3 = -4b_4, \tag{23}$$

$$b_2 = 6b_4, \tag{24}$$

$$b_1 = 16b_4, \tag{25}$$

$$b_0 = 26b_4. \tag{26}$$

To sum up, if conditions (9) are satisfied, where b_i are given by Eqs. (23)–(26), then the fundamental mode shape is expressed by Eq. (6). The fundamental natural frequency reads

$$\omega^2 = 360\frac{I}{A}\frac{b_4}{a_0L^4}. \tag{27}$$

3.2. Cantilever with linearly varying density ($m = 1$)

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0 + a_1\xi, \qquad E(\xi) = \sum_{i=0}^{5} b_i\xi^i. \tag{28}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{4} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{5} i(i-1)b_i\xi^i + 12\sum_{i=0}^{3}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{5} b_i\xi^i \\ &\quad - 48\sum_{i=0}^{4}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{5} ib_i\xi^i - kL^4(a_0 + a_1\xi)\left(6\xi^2 - 5\xi^3 + 2\xi^4\right) = 0. \end{aligned} \tag{29}$$

Eq. (29) has to be satisfied for any ξ. This requirement yields

$$24(b_0 + b_2) - 48b_1 = 0, \tag{30}$$

$$72(b_1 + b_3) - 144b_2 = 0, \tag{31}$$

$$144(b_2 + b_4) - 288b_3 - 6kL^4a_0 = 0, \tag{32}$$

$$240(b_3 + b_5) - 480b_4 + L^4(4ka_0 - 6ka_1) = 0, \tag{33}$$

$$360b_4 - 720b_5 + L^4(4ka_1 - ka_0) = 0, \tag{34}$$

$$-kL^4a_1 + 504b_5 = 0. \tag{35}$$

Coefficients b_i, where $i = \{0, 1, 2, 3, 4\}$, can be evaluated so that the compatibility of Eqs. (30)–(35) is checked;

$$b_4 = \frac{7a_0 - 18a_1}{5a_1}b_5, \tag{36}$$

$$b_3 = \frac{2(11a_1 - 14a_0)}{5a_1}b_5, \tag{37}$$

$$b_2 = \frac{2(31a_1 + 21a_0)}{5a_1}b_5, \tag{38}$$

$$b_1 = \frac{2(51a_1 + 56a_0)}{5a_1}b_5, \tag{39}$$

$$b_0 = \frac{2(71a_1 + 91a_0)}{5a_1}b_5. \tag{40}$$

We arrive at the following conclusion: if conditions (28) are satisfied, where b_i are given by Eqs. (36)–(40), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 504\frac{I}{A}\frac{b_5}{a_1L^4}. \tag{41}$$

3.3. Cantilever with parabolically varying density ($m=2$)

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0 + a_1\xi + a_2\xi^2, \qquad E(\xi) = \sum_{i=0}^{6} b_i\xi^i. \tag{42}$$

By substituting the latter expressions into Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{5} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{6} i(i-1)b_i\xi^i + 12\sum_{i=0}^{4}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{6} b_i\xi^i \\ &\quad -48\sum_{i=0}^{5}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{6} ib_i\xi^i - kL^4\left(a_0 + a_1\xi + a_2\xi^2\right)\left(6\xi^2 - 5\xi^3 + 2\xi^4\right) = 0. \end{aligned} \tag{43}$$

Eq. (43) has to be satisfied for any ξ. This requirement is equivalent to

$$24(b_0 + b_2) - 48b_1 = 0, \tag{44}$$

$$72(b_1 + b_3) - 144b_2 = 0, \tag{45}$$

$$144(b_2 + b_4) - 288b_3 - 6kL^4a_0 = 0, \tag{46}$$

$$240(b_3 + b_5) - 480b_4 + L^4(4ka_0 - 6ka_1) = 0, \tag{47}$$

$$360(b_4 + b_6) - 720b_5 + L^4(4ka_1 - ka_0 - 6ka_2) = 0, \tag{48}$$

$$504b_5 - 1008b_6 + L^4(4ka_2 - ka_1) = 0, \tag{49}$$

$$-kL^4a_2 + 504b_6 = 0. \tag{50}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3, 4, 5\}$, have to be

$$b_5 = \frac{2(2a_1 - 5a_2)}{3a_2}b_6, \tag{51}$$

$$b_4 = \frac{53a_2 - 72a_1 + 28a_0}{15a_2}b_6, \tag{52}$$

$$b_3 = \frac{4(39a_2 + 22a_1 - 28a_0)}{15a_2}b_6, \tag{53}$$

$$b_2 = \frac{259a_2 + 248a_1 + 168a_0}{15a_2}b_6, \tag{54}$$

$$b_1 = \frac{2(181a_2 + 204a_1 + 224a_0)}{15a_2}b_6, \tag{55}$$

$$b_0 = \frac{(465a_2 + 568a_1 + 728a_0)}{15a_2}b_6. \tag{56}$$

Thus, if conditions (42) are satisfied, where b_i are given by Eqs. (51)–(56), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 672\frac{I}{A}\frac{b_6}{a_2 L^4}. \tag{57}$$

3.4. Cantilever with material density represented as a cubic polynomial ($m = 3$)

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3, \qquad E(\xi) = \sum_{i=0}^{7} b_i\xi^i. \tag{58}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{6} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{7} i(i-1)b_i\xi^i + 12\sum_{i=0}^{5}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{7} b_i\xi^i \\ &\quad -48\sum_{i=0}^{6}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{7} ib_i\xi^i - kL^4\left(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3\right)\left(6\xi^2 - 5\xi^3 + 2\xi^4\right) = 0. \end{aligned} \tag{59}$$

Eq. (59) has to be satisfied for any ξ. This requirement yields

$$24(b_0 + b_2) - 48b_1 = 0, \tag{60}$$

$$72(b_1 + b_3) - 144b_2 = 0, \tag{61}$$

$$144(b_2 + b_4) - 288b_3 - 6kL^4 a_0 = 0, \tag{62}$$

$$240(b_3 + b_5) - 480b_4 + L^4(4ka_0 - 6ka_1) = 0, \tag{63}$$

$$360(b_4 + b_6) - 720b_5 + L^4(4ka_1 - ka_0 - 6ka_2) = 0, \tag{64}$$

$$504(b_5 + b_7) - 1008b_6 + L^4(4ka_2 - ka_1 - 6ka_3) = 0, \tag{65}$$

$$672b_6 - 1344b_7 + L^4(4ka_3 - ka_2) = 0, \tag{66}$$

$$-kL^4 a_3 + 864b_7 = 0. \tag{67}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3, 4, 5, 6\}$, have to be

$$b_6 = \frac{9a_2 - 22a_3}{7a_3} b_7, \tag{68}$$

$$b_5 = \frac{3(7a_3 - 10a_2 + 4a_1)}{7a_3} b_7, \tag{69}$$

$$b_4 = \frac{320a_3 + 195a_2 - 216a_1 + 84a_0}{35a_3} b_7, \tag{70}$$

$$b_3 = \frac{535a_3 + 468a_2 + 264a_1 - 336a_0}{35a_3} b_7, \tag{71}$$

$$b_2 = \frac{3(250a_3 + 259a_2 + 248a_1 + 168a_0)}{35a_3} b_7, \tag{72}$$

$$b_1 = \frac{(965a_3 + 1086a_2 + 1225a_1 + 1334a_0)}{35a_3} b_7, \tag{73}$$

$$b_0 = \frac{1180a_3 + 1395a_2 + 1704a_1 + 2184a_0}{35a_3} b_7. \tag{74}$$

In summary, if conditions (58) are satisfied, where b_i are given by Eqs. (68)–(74), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 864 \frac{I}{A} \frac{b_7}{a_3 L^4}. \tag{75}$$

3.5. Cantilever with material density represented as a higher order polynomial ($m > 3$)

Since Eq. (8) is valid for any ξ, we conclude

$$24(b_0 + b_2) - 48b_1 = 0 \quad \text{for } i = 0, \tag{76}$$

$$72(b_1 + b_3) - 144b_2 = 0 \quad \text{for } i = 1, \tag{77}$$

$$144(b_2 + b_4) - 288b_3 - 6kL^4 a_0 = 0 \quad \text{for } i = 2, \tag{78}$$

$$240(b_3 + b_5) - 480b_4 + L^4(4ka_0 - 6ka_1) = 0 \quad \text{for } i = 3, \tag{79}$$

$$12(i+1)(i+2)(b_i + b_{i+2}) - 24(i+1)(i+2)b_{i+1} + L^4(4ka_{i-3} - 6ka_{i-2} - ka_{i-4}) = 0 \quad \text{for } 4 \leqslant i \leqslant m+2, \tag{80}$$

$$12(m^2 + 9m + 42)b_{m+3} - 24(m+4)(m+5)b_{m+4} + L^4(4ka_m - ka_{m-1}) = 0 \quad \text{for } i = m+3, \tag{81}$$

$$-kL^4 a_m + 12(m^2 + 11m + 30)b_{m+4} = 0 \quad \text{for } i = m+4. \tag{82}$$

The only unknown in Eqs. (76)–(82) is the natural frequency coefficient k, yet we have $m+4$ equations. We conclude that the parameters b_i and a_i have to satisfy some auxiliary conditions in order Eqs. (76)–(82) to be compatible.

Two compatibility condition are given by Eqs. (76) and (77), leading to

$$b_0 = 2b_1 - b_2, \tag{83}$$

$$b_1 = 2b_2 - b_3. \tag{84}$$

From the other equations, several expressions for k can be found. These are determined from Eqs. (78)–(82), respectively, and are listed below:

$$k = \frac{144(b_2 + b_4) - 288b_3}{6L^4 a_0}, \tag{85}$$

$$k = \frac{240(b_3 + b_5) - 480b_4}{L^4(6a_1 - 4a_0)}, \tag{86}$$

$$k = \frac{12(i+1)(i+2)(b_i + b_{i+2}) - 24(i+1)(i+2)b_{i+1}}{L^4(a_{i-4} + 6a_{i-2} - 4a_{i-3})}, \tag{87}$$

$$k = \frac{12(m^2 + 9m + 42)b_{m+3} - 24(m+4)(m+5)b_{m+4}}{L^4(a_{m-1} - 4a_m)}, \tag{88}$$

$$k = \frac{12(m^2 + 11m + 30)b_{m+4}}{L^4 a_m}. \tag{89}$$

For meeting the compatibility requirement, all expressions for k have to be equal to each other. We consider the case when material density coefficients a_i are specified. Then problem is reduced to determining coefficients b_i so that Eqs. (85)–(89) are compatible.

Let us assume that the function $\rho(\xi)$, of the material density, and hence all a_i $(i = 0, 1, \ldots, m)$ are given. Let us observe that if b_{m+4} is specified then the expression given in Eq. (89) is the final formula for the natural frequency coefficient k. Then, Eqs. (85)–(89) allow an evaluation of remaining parameters b_i. Note that b_{m+4} and a_m have to have the same sign since the natural frequency parameter k must be positive.

From Eq. (88), we obtain

$$b_{m+3} = \frac{b_{m+4}}{a_m(m^2 + 9m + 42)}\left[(m^2 + 11m + 30)a_{m-1} - 2(m^2 + 13m + 40)a_m\right]. \tag{90}$$

Eq. (87) leads to

$$\begin{aligned} b_i = \frac{1}{(i+1)(4a_{i-2} - 6a_{i-1} - a_{i-3})} \{ & [-(i+3)a_{i-4} + 2(i+5)a_{i-3} + (2i-10)a_{i-2} - 12(i+1)a_{i-1}]b_{i+1} \\ & + [2(i+3)a_{i-4} - (7i-23)a_{i-3} + 8(i+4)a_{i-2} + 6(i+1)a_{i-1}]b_{i+2} \\ & + [-(i+3)a_{i-4} + 4(i+3)a_{i-3} + (2i-10)a_{i-2}]b_{i+3}\}, \end{aligned} \tag{91}$$

where i takes on values $\{4, 5, \ldots, m+2\}$.

Eq. (86) yields

$$b_3 = \frac{(-4a_0 + 12a_2 + a_1)b_4 + (11a_0 - 6a_2 - 14a_1)b_5 + (9a_1 - 6a_0)b_6}{a_0 + 6a_2 - 4a_1}. \tag{92}$$

From Eq. (85), we deduce

$$b_2 = \frac{(6a_1 + a_0)b_3 - (3a_1 + 8a_0)b_4 - 5a_0 b_5}{3a_1 - 2a_0}. \tag{93}$$

We conclude that for specified coefficients $a_0, \ldots, a_m$ and b_{m+4}, Eqs. (83), (84) and (90)–(93) result in the set of coefficients for the elastic modulus such that the beam has a mode shape given in Eq. (6). It is remarkable that if $a_i = a$, then the coefficients b_i do not depend on the parameter a.

To sum up, if $\rho(\xi)$ and $E(\xi)$ vary as in Eq. (4) with b_i computed via Eqs. (83), (84) and (90)–(93), the fundamental mode shape of a beam is given by Eq. (6) and the fundamental natural frequency squared reads

$$\omega^2 = \frac{12I}{AL^4}\left(m^2 + 11m + 30\right)\frac{b_{m+4}}{a_m}. \tag{94}$$

As we have established, in order for the closed-form solution to be obtainable, it is sufficient that (1) all a_i coefficients and (2) the coefficient b_{m+4} be specified. These requirements are not necessary: one can assume that all a_i coefficients are given and instead of the coefficient b_{m+4} any other b_j coefficient ($j \neq m+4$) is specified. If this is the case, then from Eq. (87) one expresses b_{j+1} via b_j and k; substitution into subsequent equations allows us to express b_{m+2}, b_{m+3}, b_{m+4} via b_i; analogously, substitution of b_i into Eqs. (86)–(88) yields sought exact solutions.

Although the natural frequency expressions for uniform density in Eq. (27), for linearly varying density in Eq. (41), for parabolically varying density in Eq. (57) and cubic varying density in Eq. (75), are derived separately from the case $m > 3$, all these equations follow from Eq. (94) by substituting appropriate values for m. Hence, Eq. (94) is the final formula for any integer value m.

4. Beam that is clamped at both ends

The beam is clamped at $\xi = 0$ and $\xi = 1$. The boundary conditions are

$$W(\xi) = 0, \quad \frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi} = 0 \qquad \text{at } \xi = 0, 1. \tag{95}$$

A simplest polynomial function that satisfies boundary conditions in Eq. (95) is given by

$$W(\xi) = w_1\left(\xi^2 - 2\xi^3 + \xi^4\right). \tag{96}$$

By substituting the expressions for $E(\xi)$, $\rho(\xi)$, $W(\xi)$ in Eq. (3), we obtain

$$w_1\left[\sum_{i=2}^{m+4} i(i-1)b_i\xi^{i-2}\left(12\xi^2 - 12\xi + 2\right) + \sum_{i=0}^{m+4} 24b_i\xi^i + 2\sum_{i=1}^{m+4} ib_i\xi^{i-1}(24\xi - 12) - kL^4\sum_{i=0}^{m} a_i\xi^i\left(\xi^4 - 2\xi^3 + \xi^2\right)\right] = 0. \tag{97}$$

The latter expression can be cast in the following form:

$$-12\sum_{i=1}^{m+3} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{m+4} i(i-1)b_i\xi^i + 2\sum_{i=0}^{m+2}(i+2)(i+1)b_{i+2}\xi^i + 24\sum_{i=0}^{m+4} b_i\xi^i - 24\sum_{i=0}^{m+3}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{m+4} ib_i\xi^i - kL^4\sum_{i=2}^{m+2} a_{i-2}\xi^i + 2kL^4\sum_{i=3}^{m+3} a_{i-3}\xi^i - kL^4\sum_{i=4}^{m+4} a_{i-4}\xi^i = 0. \tag{98}$$

Since Eq. (98) has to be satisfied for any ξ, we arrive at the following relations:

$$24b_0 + 4b_2 - 24b_1 = 0 \quad \text{for } i = 0, \tag{99}$$

$$72b_1 + 12b_3 - 72b_2 = 0 \quad \text{for } i = 1, \tag{100}$$

$$144b_2 + 24b_4 - 144b_3 - kL^4a_0 = 0 \quad \text{for } i = 2, \tag{101}$$

$$240b_3 + 40b_5 - 240b_4 + L^4(2ka_0 - ka_1) = 0 \quad \text{for } i = 3, \tag{102}$$

$$12(i+1)(i+2)b_i + 2(i+1)(i+2)b_{i+2} - 12(i+1)(i+2)b_{i+1} + L^4(2ka_{i-3} - ka_{i-2} - ka_{i-4}) = 0 \quad \text{for } 4 \leqslant i \leqslant m+2, \tag{103}$$

$$12(m+4)(m+5)b_{m+3} - 12(m+4)(m+5)b_{m+4} + L^4(2ka_m - ka_{m-1}) = 0 \quad \text{for } i = m+3, \tag{104}$$

$$-kL^4 a_m + 12(m^2 + 11m + 30)b_{m+4} = 0 \quad \text{for } i = m+4. \tag{105}$$

It should be borne in mind that Eqs. (99)–(105) are valid only if $m > 3$. For cases that satisfy the inequality $m \leqslant 4$, the reader is referred to Appendix A. Note also that the Eqs. (99)–(105) have a recursive form, as do Eqs. (76)–(82) for the cantilever.

Two compatibility conditions are immediately detected for Eqs. (99) and (100), resulting in

$$b_0 = b_1 - b_2/6, \tag{106}$$

$$b_1 = b_2 - b_3/6. \tag{107}$$

From other equations, several expressions for k can be found. These are determined from Eqs. (101)–(105) respectively. The alternative analytical formulas for k read

$$k = \frac{144(b_2 - b_3) - 24b_4}{L^4 a_0}, \tag{108}$$

$$k = \frac{240(b_3 - b_4) - 40b_5}{L^4(a_1 - 2a_0)}, \tag{109}$$

$$k = \frac{12(i+1)(i+2)(b_i - b_{i+1}) - 2(i+1)(i+2)b_{i+2}}{L^4(a_{i-4} + a_{i-2} - 2a_{i-3})}, \tag{110}$$

$$k = \frac{12(m+4)(m+5)(b_{m+3} - b_{m+4})}{L^4(a_{m-1} - 2a_m)}, \tag{111}$$

$$k = \frac{12(m^2 + 11m + 30)b_{m+4}}{L^4 a_m}. \tag{112}$$

Eqs. (108)–(112) then allow an evaluation of remaining parameters b_i. It is noteworthy that b_{m+4} and a_m have to have the same sign due to the positivity of k. From Eq. (111), we get

$$b_{m+3} = \frac{b_{m+4}}{a_m(m+4)(m+5)}\left[(m^2 + 11m + 30)a_{m-1} - (m^2 + 13m + 40)a_m\right]. \tag{113}$$

Eq. (110) yields

$$\begin{aligned} b_i = \frac{1}{6(i+1)(2a_{i-2} - a_{i-1} - a_{i-3})} \{ & [-6(i+3)a_{i-4} + 6(i+5)a_{i-3} + 6(i-1)a_{i-2} - 6(i+1)a_{i-1}]b_{i+1} \\ & + [6(i+3)a_{i-4} - (37i+13)a_{i-3} + 4(2i+5)a_{i-2} - (i+1)a_{i-1}]b_{i+2} \\ & + [(i+3)a_{i-4} - 2(i+3)a_{i-3} + (i+3)a_{i-2}]b_{i+3} \}, \end{aligned} \tag{114}$$

where i belongs to the set $\{4, 5, \dots, m+2\}$. Eq. (109) leads to

$$b_3 = \frac{6(4a_0 + a_1 - 2a_2)b_4 - (38a_0 - 2a_2 - 22a_1)b_5 - 3(2a_0 - a_1)b_6}{12(2a_1 - a_0 - a_2)}. \tag{115}$$

From Eq. (108), we obtain

$$b_2 = \frac{(18a_1 - 6a_0)b_3 + (3a_1 - 36a_0)b_4 - 5a_0 b_5}{18(a_1 - 2a_0)}. \tag{116}$$

Therefore, for specified coefficients $a_0, a_1, \dots, a_m$ and b_{m+4}, Eqs. (106), (107) and (113)–(116) lead to the set coefficients in elastic modulus such that the beam possesses mode shape given in Eq. (96). In perfect analysis with the cantilever, if $a_i = a$, then coefficients b_i do not depend on the parameter a. We conclude that, if $\rho(\xi)$ and $E(\xi)$ vary as in Eq. (4), with attendant b_i computed via Eqs. (106), (107) and (113)–(116), the fundamental mode shape of the beam is governed by Eq. (96), and the fundamental natural frequency squared reads

$$\omega^2 = \frac{12I}{AL^4}\left(m^2 + 11m + 30\right)\frac{b_{m+4}}{a_m}. \tag{117}$$

5. Beam clamped at one end and simply supported at the other

Consider now the beam that is clamped at $\xi = 0$ and simply supported at $\xi = 1$. The boundary conditions are

$$\begin{aligned} &W(\xi) = 0, \quad \frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi} = 0 \qquad \text{at } \xi = 0, \\ &W(\xi) = 0, \quad EI\frac{\mathrm{d}^2 W(\xi)}{\mathrm{d}\xi^2} = 0 \qquad \text{at } \xi = 1. \end{aligned} \tag{118}$$

A simplest polynomial function that satisfies boundary conditions in Eq. (118), and does not have a nodal point in the interval (0,1) is given by

$$W(\xi) = w_1\left(3\xi^2 - 5\xi^3 + 2\xi^4\right). \tag{119}$$

By substituting the expressions for $E(\xi)$, $\rho(\xi)$, $W(\xi)$ in Eq. (3), we obtain

$$\begin{aligned} w_1\Bigg[&\sum_{i=0}^{m+4} i(i-1)b_i\xi^{i-2}\left(24\xi^2 - 30\xi + 6\right) + \sum_{i=0}^{m+4} 48b_i\xi^i + 2\sum_{i=1}^{m+4} ib_i\xi^{i-1}(48\xi - 30) \\ &- kL^4\sum_{i=0}^{m} a_i\xi^i\left(2\xi^4 - 5\xi^3 + 3\xi^2\right)\Bigg] = 0. \end{aligned} \tag{120}$$

Eq. (120) can be re-written as follows:

$$\begin{aligned} &-30\sum_{i=1}^{m+3} i(i+1)b_{i+1}\xi^i + 24\sum_{i=2}^{m+4} i(i-1)b_i\xi^i + 6\sum_{i=0}^{m+2}(i+2)(i+1)b_{i+2}\xi^i + 48\sum_{i=0}^{m+4} b_i\xi^i \\ &- 60\sum_{i=0}^{m+3}(i+1)b_{i+1}\xi^i + 96\sum_{i=1}^{m+4} ib_i\xi^i - 3kL^4\sum_{i=2}^{m+2} a_{i-2}\xi^i + 5kL^4\sum_{i=3}^{m+3} a_{i-3}\xi^i - 2kL^4\sum_{i=4}^{m+4} a_{i-4}\xi^i = 0. \end{aligned} \tag{121}$$

Since Eq. (121) has to be satisfied for any ξ. We conclude that

$$48b_0 - 60b_1 + 12b_2 = 0 \quad \text{for } i = 0, \tag{122}$$

$$144b_1 - 180b_2 + 36b_3 = 0 \quad \text{for } i = 1, \tag{123}$$

$$288b_2 - 360b_3 + 72b_4 - 3kL^4 a_0 = 0 \quad \text{for } i = 2, \tag{124}$$

$$480b_3 - 600b_4 + 120b_5 + L^4(5ka_0 - 3ka_1) = 0 \quad \text{for } i = 3, \tag{125}$$

$$24(i+1)(i+2)b_i - 30(i+1)(i+2)b_{i+1} + 6(i+1)(i+2)b_{i+2} + L^4(5ka_{i-3} - 3ka_{i-2} - 2ka_{i-4}) = 0 \quad \text{for } 4 \leqslant i \leqslant m+2, \tag{126}$$

$$24(m+4)(m+5)b_{m+3} - 30(m+4)(m+5)b_{m+4} + L^4(5ka_m - 2ka_{m-1}) = 0 \quad \text{for } i = m+3, \tag{127}$$

$$-kL^4 a_m + 12(m^2 + 11m + 30)b_{m+4} = 0 \quad \text{for } i = m+4. \tag{128}$$

It should be borne in mind that Eqs. (122)–(128) are valid only if $m > 3$. For cases that satisfy the inequality $m \leqslant 3$, the reader is referred to Appendix B. Note also that the Eqs. (122)–(128) have a recursive form.

Two compatibility conditions are given by Eqs. (122) and (123), with

$$b_0 = \frac{5b_1 - b_2}{4}, \tag{129}$$

$$b_1 = \frac{5b_2 - b_3}{4}. \tag{130}$$

From the other equations, several expressions for k can be found. These are determined from Eqs. (124)–(128) respectively, are listed below:

$$k = \frac{96b_2 - 120b_3 + 24b_4}{L^4 a_0}, \tag{131}$$

$$k = \frac{480b_3 - 600b_4 + 120b_5}{L^4(3a_1 - 5a_0)}, \tag{132}$$

$$k = \frac{6(i+1)(i+2)(4b_i - 5b_{i+1} + b_{i+2})}{L^4(2a_{i-4} + 3a_{i-2} - 5a_{i-3})}, \tag{133}$$

$$k = \frac{6(m+4)(m+5)(4b_{m+3} - 5b_{m+4})}{L^4(2a_{m-1} - 5a_m)}, \tag{134}$$

$$k = \frac{12(m^2 + 11m + 30)b_{m+4}}{L^4 a_m}. \tag{135}$$

Then, Eqs. (131)–(135) allow an evaluation of remaining parameters b_i. Note that b_{m+4} and a_m have to have the same sign due to the positivity of k. From Eq. (134), we obtain

$$b_{m+3} = \frac{b_{m+4}}{4a_m(m+4)}[4(m+6)a_{m-1} - 5(m+8)a_m]. \tag{136}$$

Eq. (133) yields

$$b_i = \frac{1}{4(i+1)(5a_{i-2} - 3a_{i-1} - 2a_{i-3})}\{[-8(i+3)a_{i-4} + 10(i+5)a_{i-3} + (13i-11)a_{i-2} - 15(i+1)a_{i-1}]b_{i+1} + [10(i+3)a_{i-4} - (23i+73)a_{i-3} + 10(i+4)a_{i-2} + 3(i+1)a_{i-1}]b_{i+2} + [-2(i+3)a_{i-4} + 5(i+3)a_{i-3} - 3(i+3)a_{i-2}]b_{i+3}\}, \tag{137}$$

where i belongs to the set $\{4, 5, \ldots, m+2\}$. Eq. (132) results in

$$b_3 = \frac{(40a_0 - 30a_2 + 14a_1)b_4 + (-71a_0 + 6a_2 + 35a_1)b_5 + (-9a_1 + 15a_0)b_6}{8(5a_1 - 3a_2 - 2a_0)}. \tag{138}$$

From Eq. (131), we obtain

$$b_2 = \frac{5(5a_1 - a_0)b_3 - (3a_1 + 20a_0)b_4 + 5a_0b_5}{4(3a_1 - 5a_0)}. \tag{139}$$

Thus, for specified coefficients $a_0, a_1, \ldots, a_m$ and b_{m+4}, Eqs. (129), (130) and (134)–(139) lead to the set of coefficients in elastic modulus such that the beam possesses mode shape given in Eq. (119). Note that if $a_i = a$, then coefficients b_i do not depend on the parameter a.

The results can be summarized as follows: if $\rho(\xi)$ and $E(\xi)$ vary as in Eq. (4) with b_i computed via Eqs. (129), (130) and (134)–(139), the fundamental mode shape of a beam is given by Eq. (119), and the fundamental natural frequency squared reads

$$\omega^2 = \frac{12I}{AL^4}(m^2 + 11m + 30)\frac{b_{m+4}}{a_m}. \tag{140}$$

6. Random beams with deterministic frequencies

As is seen in Eqs. (94), (117) and (140), the fundamental natural frequency depends only upon terminal coefficients a_m and b_{m+4}. If either of these coefficients is random, so is the natural frequency. In latter case one can pose a problem of the reliability evaluation, the reliability being defined as the probability that the natural frequency does not exceed any pre-selected value. Such an analysis, for the beams that were simply supported at their ends was conducted by Candan and Elishakoff (2000).

Yet, Eqs. (94), (117) and (140) for beams with three different boundary conditions lead to a remarkable conclusion: If the material density coefficients $a_0, a_1, \ldots, a_{m-1}$, and the elastic modulus coefficients $b_0, b_1, \ldots, b_{m+3}$ are *random*, but the quantities a_m and b_{m+4} are *deterministic*, the natural frequency is a *deterministic* variable too. Thus, although beam is random, its fundamental frequency is *deterministic*. The present writers do not know of any other study that reports an analogous occurrence. The closest one is the study by Fraser and Budiansky (1969) which dealt with buckling of elastic beams on non-linear elastic foundation; the beam possessed with random initial imperfection – deviation from the straight line – with given autocorrelation function. The beams were treated as having infinite length and the initial imperfections constituted ergodic random field. Fraser and Budiansky (1969) used approximate technique which resulted in deterministic buckling loads. The buckling load was defined as a maximum axial load the beam could support. Several other studies followed (see bibliography of Amazigo, 1976) which utilized different approximate analyses. Amazigo (1976) ascribed this seemingly paradoxical behavior to the property of ergodicity that was postulated for the random initial imperfection.

Elishakoff (1979) re-examined the Fraser–Budiansky problem, for finite beams on the non-linear elastic foundations. Monte Carlo method was utilized in conjunction with developing a special procedure for solving a non-linear boundary-value problem for each realization of the random beam. In *finite* beams, the buckling loads did not turn out to be deterministic. Yet, with the increase of the length, the coefficient of variation of the buckling load was showed to be a *decreasing* function.

How can one detect that a complex structure may possess deterministic eigenvalues? Such structures obviously are analyzed by approximate analytical and/or numerical techniques. In order to answer this question, we simulate the realistic situation and apply the Monte Carlo method to check the validity of the main conclusion of this study, namely, that random beams may have deterministic frequencies.

The particular case considered hereinafter is $m = 2$. Coefficients a_0 and a_1 were taken to be exponentially distributed independent random variables. One thousand and eighty nine realizations of beams were simulated. For each realization of coefficients a_0 and a_1 the appropriate coefficients $b_0, \ldots, b_5$ were evaluated. For simplicity the coefficients $a_m = a_2$ and $b_{m+4} = b_6$ were fixed at unity. For each realization of the beam the finite element method was applied. For one of the simulated beams, with material density coefficients are $a_0 = 0.557602$, $a_1 = 0.387297$, $a_2 = 1$. The associated elastic modulus coefficients are given in the Tables 3–6, depending on the boundary condition. The convergence of the finite-element method is illustrated in Table 1 with exact solution for the natural frequency coefficient being $k = 672$. The percentagewise errors from the exact solution is listed in Table 2. For the subsequent calculations, the number of elements was taken to equal four, with maximum error, that for the clamped–clamped beam, being only 0.41%. This is in agreement with the observation by Gupta and Rao (1978) concluding that "the finite-element procedure developed for the eigenvalue analysis of doubly tapered and twisted Timoshenko beams has been found to give reasonably accurate results even with four finite elements." Sample calculations for ten realizations of the random beam are listed in Tables 3–6 for various boundary conditions. For the case of the beams simply supported at both ends, Table 4 also lists the results obtainable from the single-term Boobnov–Galerkin method with sinusoidal comparison function $\sin \pi\xi$. It can be observed from Table 3, the sample frequencies are concentrated around the exact solution $k = 672$ none exceeding the value 673. The same occurs for the clamped–free beams (Table 4). For clamped-simply supported beams none of the random frequencies in Table 5 exceeds 674, whereas for clamped–clamped beam all frequencies in Table 6 are below 675.

Results of the Monte Carlo simulation for 1089 sample beams is given in Table 7. At this size of the sample, according to Massey (1951), at the level of significance of 0.01 the maximum absolute difference between exact and empirical reliabilities is smaller than $1.63/\sqrt{1089} = 0.049$. It lists mean values, standard

Table 1
Convergence of the natural frequency squared for differents boundary conditions[a]

Number of elements	Simply supported	Clamped supported	Clamped–clamped	Clamped-free
1	791.789673	1146.29273		696.96113
2	678.667018	691.102962	717.006005	673.551345
3	673.321172	675.750735	680.733099	672.307085
4	672.418974	673.188072	674.761629	672.097261
5	672.171821	672.487131	673.131839	672.039859
6	672.082919	672.235081	672.546128	672.019228
7	672.044777	672.126948	672.294903	672.01038
8	672.026255	672.074438	672.172916	672.006086
9	672.016394	672.046481	672.107973	672.0038
10	672.010758	672.030501	672.070852	672.002493

[a] $a[0] = 0.557602$, $a[1] = 0.387297$, and $a[2] = 1$.

Table 2
Percentagewise error

Number of elements	Simply supported	Clamped supported	Clamped–clamped	Clamped–free
1	17.8258442	70.5792753		3.714453869
2	0.992115774	2.842702679	6.697322173	0.230854911
3	0.196602976	0.558145089	1.299568304	0.045697173
4	0.062347321	0.176796429	0.410956696	0.014473363
5	0.025568601	0.072489732	0.168428423	0.005931399
6	0.012339137	0.034982292	0.081269048	0.00286131
7	0.006663244	0.018891071	0.043884375	0.001544643
8	0.003906994	0.011077083	0.025731548	0.000905655
9	0.002439583	0.006916815	0.016067411	0.000565476
10	0.001600893	0.004538839	0.010543452	0.000370982

deviation and the coefficient of variation for the natural frequencies. It is seen for all four sets of boundary conditions, the standard deviation is much smaller than the mean natural frequency. Resulting coefficients of variation all are less than 10^{-5}. Extreme smallness of the coefficient of variation supports our theoretical finding that the natural frequency constitutes a deterministic quantity.

Likewise, it appears that if the results of the Monte Carlo simulation of a complex structure exhibit small coefficients of variation for the *eigenvalues* irrespective of the moderate a large coefficients of variation of the *input* stochastic quantities, one is facing the phenomenon uncovered in this study, namely the random structures possessing the deterministic eigenvalues.

The difference with the paper by Fraser and Budiansky (1969) lies in the fact that the deterministic property of the buckling loads was possibly due to the ergodicity of the input random fields (Amazigo, 1976) expanding from minus infinity to plus infinity, combined with approximate analysis. Here, we do not use ergodicity assumption, the beams have finite length and the results are obtained in the closed form.

7. Conclusions

For the three sets of boundary conditions the closed-form solutions have been derived for the mode shapes and natural frequencies of non-homogeneous beams. Following conclusions have been reached:

(1) Non-homogeneous beams may possess the natural mode that is coincident with the static deflection of the associated uniform beam under uniformly distributed load.

(2) The fundamental frequencies in all three cases coincide with each other, as the comparison of Eqs. (94), (117) and (140) reveals. The remaining case of the inhomogeneous beam that is simply-supported at its both ends was studied by Candan and Elishakoff (2000). There too, the beam turned out to possess the fundamental frequency given in Eqs. (94), (117) and (140).

(3) Although the expressions for fundamental frequencies of inhomogeneous beams with four different boundary conditions coalesce, the beam's characteristics in each case are *different*. Namely, although they share the same material density variation as in Eq. (4), the b_i coefficients in the elastic modulus variation *differ*. This leads to the interesting conclusion that the beams with *different* elastic modulus variation may have the same natural frequency, although the beams are under differing boundary conditions. This conclusion may at first glance appear to be counterintuitive. Indeed, if one anticipates that the fundamental frequency of the clamped–clamped beam must be greater than its counterpart for the beam that is simply supported at its both ends. Yet, it must be borne in mind that in the cases in our consideration the beams'

Table 3
Sample calculations for natural frequencies of simply supported beams

Coefficients $a[j]$	Coefficients $b[j]$	k: Natural frequency coefficient	
		Boobnov–Galerkin method	FEM
$a[0] = 0.557602$ $a[1] = 0.387297$ $a[2] = 1.000000$	$b[0] = 5.133779$, $b[1] = 5.133779$ $b[2] = -0.070508$, $b[3] = -1.87789$ $b[4] = -1.555322$, $b[5] = -1.150270$ $b[6] = 1.000000$	672.907049	672.418974
$a[0] = 0.421035$ $a[1] = 0.642941$ $a[2] = 1.000000$	$b[0] = 4.948464$, $b[1] = 4.948464$ $b[2] = 1.018801$, $b[3] = -1.981591$ $b[4] = -2.423793$, $b[5] = -0.809412$ $b[6] = 1.000000$	672.907092	672.418932
$a[0] = 0.817484$ $a[1] = 1.373393$ $a[2] = 1.000000$	$b[0] = 8.824267$, $b[1] = 8.824267$ $b[2] = 1.194419$, $b[3] = -5.214750$ $b[4] = -3.436841$, $b[5] = 0.164524$ $b[6] = 1.000000$	672.904906	672.418277
$a[0] = 0.032587$ $a[1] = 1.283644$ $a[2] = 1.000000$	$b[0] = 4.225417$, $b[1] = 4.225417$ $b[2] = 3.921267$, $b[3] = -2.069073$ $b[4] = -4.686583$, $b[5] = 0.044859$ $b[6] = 1.000000$	672.907457	672.418818
$a[0] = 0.927113$ $a[1] = 1.368769$ $a[2] = 1.000000$	$b[0] = 9.427710$, $b[1] = 9.427710$ $b[2] = 0.774654$, $b[3] = -5.612935$ $b[4] = -3.221101$, $b[5] = 0.158359$ $b[6] = 1.000000$	672.90474	672.418238
$a[0] = 0.300851$ $a[1] = 0.371173$ $a[2] = 1.000000$	$b[0] = 3.659425$, $b[1] = 3.659425$ $b[2] = 0.851482$, $b[3] = -0.880659$ $b[4] = -1.995894$, $b[5] = -1.171769$ $b[6] = 1.000000$	672.9088	672.419444
$a[0] = 1.595434$ $a[1] = 0.039957$ $a[2] = 1.000000$	$b[0] = 10.158333$, $b[1] = 10.158333$ $b[2] = -4.732383$, $b[3] = -4.918851$ $b[4] = 1.215579$, $b[5] = -1.613390$ $b[6] = 1.000000$	672.904734	672.41833
$a[0] = 0.092588$ $a[1] = 3.231425$ $a[2] = 1.000000$	$b[0] = 8.976387$, $b[1] = 8.976387$ $b[2] = 8.112237$, $b[3] = -6.967747$ $b[4] = -9.249257$, $b[5] = 2.641900$ $b[6] = 1.000000$	672.904605	672.417972
$a[0] = 1.676221$ $a[1] = 0.604673$ $a[2] = 1.000000$	$b[0] = 11.890765$, $b[1] = 11.890765$ $b[2] = -3.753968$, $b[3] = -6.575775$ $b[4] = 0.011065$, $b[5] = -0.860436$ $b[6] = 1.000000$	672.904289	672.418169
$a[0] = 1.032913$ $a[1] = 2.533571$ $a[2] = 1.000000$	$b[0] = 12.660406$, $b[1] = 12.660406$ $b[2] = 3.019887$, $b[3] = -8.803445$ $b[4] = -5.819134$, $b[5] = 1.711428$ $b[6] = 1.000000$	672.904	672.417973

characteristics are different: The clamped–clamped beam and the simply supported beam have different expressions for the elastic modulus. Also, as seen from Tables 4–6, the natural frequency of the beams under different boundary conditions are fixed to be equal to each other by making $a_2 = 1$ and $b_6 = 1$ where

Table 4
Sample calculations for natural frequencies of clamped–free beams

Coefficients $a[j]$	Coefficients $b[j]$	Natural frequency coefficient k by the FEM
$a[0] = 0.557602$ $a[1] = 0.387297$ $a[2] = 1.000000$	$b[0] = 72.727949$, $b[1] = 51.321537$ $b[2] = 29.915125$, $b[3] = 8.508713$ $b[4] = 2.715164$, $b[5] = -2.816937$ $b[6] = 1.000000$	672.097261
$a[0] = 0.421035$ $a[1] = 0.642941$ $a[2] = 1.000000$	$b[0] = 75.780285$, $b[1] = 54.196254$ $b[2] = 32.612222$, $b[3] = 11.028191$ $b[4] = 1.233148$, $b[5] = -2.476078$ $b[6] = 1.000000$	672.097963
$a[0] = 0.817484$ $a[1] = 1.373393$ $a[2] = 1.000000$	$b[0] = 122.681037$, $b[1] = 85.905145$ $b[2] = 49.129253$, $b[3] = 12.353362$ $b[4] = -1.532985$, $b[5] = -1.502142$ $b[6] = 1.000000$	672.096366
$a[0] = 0.032587$ $a[1] = 1.283644$ $a[2] = 1.000000$	$b[0] = 81.188912$, $b[1] = 60.021739$ $b[2] = 38.854566$, $b[3] = 17.687394$ $b[4] = -2.567330$, $b[5] = -1.621808$ $b[6] = 1.000000$	672.099769
$a[0] = 0.927113$ $a[1] = 1.368769$ $a[2] = 1.000000$	$b[0] = 127.826609$, $b[1] = 89.053628$ $b[2] = 50.280647$, $b[3] = 11.507667$ $b[4] = -1.306147$, $b[5] = -1.508308$ $b[6] = 1.000000$	672.096056
$a[0] = 0.300851$ $a[1] = 0.371173$ $a[2] = 1.000000$	$b[0] = 59.656390$, $b[1] = 43.214658$ $b[2] = 26.772926$, $b[3] = 10.331195$ $b[4] = 2.313291$, $b[5] = -2.838436$ $b[6] = 1.000000$	672.098978
$a[0] = 1.595434$ $a[1] = 0.039957$ $a[2] = 1.000000$	$b[0] = 109.944777$, $b[1] = 72.870466$ $b[2] = 35.796155$, $b[3] = -1.278156$ $b[4] = 6.319681$, $b[5] = -3.280057$ $b[6] = 1.000000$	672.093444
$a[0] = 0.092588$ $a[1] = 3.231425$ $a[2] = 1.000000$	$b[0] = 157.856878$, $b[1] = 114.793376$ $b[2] = 71.729875$, $b[3] = 28.666374$ $b[4] = -11.804677$, $b[5] = 0.975233$ $b[6] = 1.000000$	672.098338
$a[0] = 1.676221$ $a[1] = 0.604673$ $a[2] = 1.000000$	$b[0] = 135.249566$, $b[1] = 90.643587$ $b[2] = 46.037608$, $b[3] = 1.431629$ $b[4] = 3.759849$, $b[5] = -2.527103$ $b[6] = 1.000000$	672.093899
$a[0] = 1.032913$ $a[1] = 2.533571$ $a[2] = 1.000000$	$b[0] = 177.068593$, $b[1] = 123.896129$ $b[2] = 70.723666$, $b[3] = 17.551202$ $b[4] = -6.699704$, $b[5] = 0.044762$ $b[6] = 1.000000$	672.096116

$\omega^2 = 672Ib_6/Aa_2L^4$. It is obvious that other coefficients a_0, a_1, b_0, b_1, b_2, b_3, b_4 and b_5 are going to be different for each boundary condition. This circumstance will result in getting *different* elastic modulus variations for the inhomogeneous beams under *different* boundary conditions.

Table 5
Sample calculations for natural frequencies of clamped supported beams

Coefficients $a[j]$	Coefficients $b[j]$	Natural frequency coefficient k by the FEM
$a[0] = 0.557602$ $a[1] = 0.387297$ $a[2] = 1.000000$	$b[0] = 2.386304$, $b[1] = 2.172488$ $b[2] = 1.317223$, $b[3] = -2.103834$ $b[4] = -0.175201$, $b[5] = -1.566937$ $b[6] = 1.000000$	673.188072
$a[0] = 0.421035$ $a[1] = 0.642941$ $a[2] = 1.000000$	$b[0] = 2.352934$, $b[1] = 2.172475$ $b[2] = 1.450641$, $b[3] = -1.436697$ $b[4] = -1.197058$, $b[5] = -1.226078$ $b[6] = 1.000000$	673.190547
$a[0] = 0.817484$ $a[1] = 1.373393$ $a[2] = 1.000000$	$b[0] = 4.127546$, $b[1] = 3.778540$ $b[2] = 2.382518$, $b[3] = -3.201570$ $b[4] = -2.648377$, $b[5] = -0.252142$ $b[6] = 1.000000$	673.18739
$a[0] = 0.032587$ $a[1] = 1.283644$ $a[2] = 1.000000$	$b[0] = 2.159529$, $b[1] = 2.078161$ $b[2] = 1.752691$, $b[3] = 0.450809$ $b[4] = -3.844270$, $b[5] = -0.371808$ $b[6] = 1.000000$	673.197488
$a[0] = 0.927113$ $a[1] = 1.368769$ $a[2] = 1.000000$	$b[0] = 4.382875$, $b[1] = 3.997323$ $b[2] = 2.455116$, $b[3] = -3.713713$ $b[4] = -2.429862$, $b[5] = -0.258308$ $b[6] = 1.000000$	673.186588
$a[0] = 0.300851$ $a[1] = 0.371173$ $a[2] = 1.000000$	$b[0] = 1.757614$, $b[1] = 1.630702$ $b[2] = 1.123053$, $b[3] = -0.907543$ $b[4] = -0.606097$, $b[5] = -1.588436$ $b[6] = 1.000000$	673.192617
$a[0] = 1.595434$ $a[1] = 0.039957$ $a[2] = 1.000000$	$b[0] = 4.457584$, $b[1] = 3.912576$ $b[2] = 1.732547$, $b[3] = -6.987571$ $b[4] = 2.804104$, $b[5] = -2.030057$ $b[6] = 1.000000$	673.179294
$a[0] = 0.092588$ $a[1] = 3.231425$ $a[2] = 1.000000$	$b[0] = 4.521305$, $b[1] = 4.324959$ $b[2] = 3.539577$, $b[3] = 0.398050$ $b[4] = -9.575612$, $b[5] = 2.225233$ $b[6] = 1.000000$	673.194604
$a[0] = 1.676221$ $a[1] = 0.604673$ $a[2] = 1.000000$	$b[0] = 5.293016$, $b[1] = 4.693411$ $b[2] = 2.294989$, $b[3] = -7.298696$ $b[4] = 1.260761$, $b[5] = -1.277103$ $b[6] = 1.000000$	673.180924
$a[0] = 1.032913$ $a[1] = 2.533571$ $a[2] = 1.000000$	$b[0] = 5.961455$, $b[1] = 5.483693$ $b[2] = 3.572644$, $b[3] = -4.071551$ $b[4] = -5.726776$, $b[5] = 1.294762$ $b[6] = 1.000000$	673.187544
$a[0] = 3.017028$ $a[1] = 0.158394$ $a[2] = 1.000000$	$b[0] = 7.971769$, $b[1] = 6.944167$ $b[2] = 2.833762$, $b[3] = -13.607860$ $b[4] = 5.102438$, $b[5] = -1.872142$ $b[6] = 1.000000$	673.176665

Table 6
Sample calculations for natural frequencies of clamped–clamped beams

Coefficients $a[j]$	Coefficients $b[j]$	Natural frequency coefficient k by the FEM
$a[0] = 0.557602$ $a[1] = 0.387297$ $a[2] = 1.000000$	$b[0] = 1.013858$, $b[1] = 1.159905$ $b[2] = 0.876280$, $b[3] = -1.701751$ $b[4] = 0.144678$, $b[5] = -1.150270$ $b[6] = 1.000000$	674.761629
$a[0] = 0.421035$ $a[1] = 0.642941$ $a[2] = 1.000000$	$b[0] = 0.973584$, $b[1] = 1.130086$ $b[2] = 0.939010$, $b[3] = -1.146453$ $b[4] = -0.723793$, $b[5] = -0.809412$ $b[6] = 1.000000$	674.76113
$a[0] = 0.817484$ $a[1] = 1.373393$ $a[2] = 1.000000$	$b[0] = 1.755885$, $b[1] = 2.023840$ $b[2] = 1.607728$, $b[3] = -2.496669$ $b[4] = -1.736841$, $b[5] = 0.164524$ $b[6] = 1.000000$	674.764592
$a[0] = 0.032587$ $a[1] = 1.283644$ $a[2] = 1.000000$	$b[0] = 0.819969$, $b[1] = 0.997884$ $b[2] = 1.067493$, $b[3] = 0.417655$ $b[4] = -2.986583$, $b[5] = 0.044859$ $b[6] = 1.000000$	674.758823
$a[0] = 0.927113$ $a[1] = 1.368769$ $a[2] = 1.000000$	$b[0] = 1.878854$, $b[1] = 2.157733$ $b[2] = 1.673270$, $b[3] = -2.906774$ $b[4] = -1.521101$, $b[5] = 0.158359$ $b[6] = 1.000000$	674.764933
$a[0] = 0.300851$ $a[1] = 0.371173$ $a[2] = 1.000000$	$b[0] = 0.713686$, $b[1] = 0.831554$ $b[2] = 0.707207$, $b[3] = -0.746080$ $b[4] = -0.295894$, $b[5] = -1.171769$ $b[6] = 1.000000$	674.758412
$a[0] = 1.595434$ $a[1] = 0.039957$ $a[2] = 1.000000$	$b[0] = 2.040809$, $b[1] = 2.261035$ $b[2] = 1.321356$, $b[3] = -5.638072$ $b[4] = 2.915579$, $b[5] = -1.613390$ $b[6] = 1.000000$	674.765655
$a[0] = 0.092588$ $a[1] = 3.231425$ $a[2] = 1.000000$	$b[0] = 1.768522$, $b[1] = 2.140223$ $b[2] = 2.230211$, $b[3] = 0.539927$ $b[4] = -7.549257$, $b[5] = 2.641900$ $b[6] = 1.000000$	674.763433
$a[0] = 1.676221$ $a[1] = 0.604673$ $a[2] = 1.000000$	$b[0] = 2.388134$, $b[1] = 2.671119$ $b[2] = 1.697904$, $b[3] = -5.839285$ $b[4] = 1.711065$, $b[5] = -0.860436$ $b[6] = 1.000000$	674.766146
$a[0] = 1.032913$ $a[1] = 2.533571$ $a[2] = 1.000000$	$b[0] = 2.525862$, $b[1] = 2.927879$ $b[2] = 2.412098$, $b[3] = -3.094684$ $b[4] = -4.119134$, $b[5] = 1.711428$ $b[6] = 1.000000$	674.765766
$a[0] = 3.017028$ $a[1] = 0.158394$ $a[2] = 1.000000$	$b[0] = 3.715995$, $b[1] = 4.095051$ $b[2] = 2.274334$, $b[3] = -10.924301$ $b[4] = 5.284974$, $b[5] = -1.455475$ $b[6] = 1.000000$	674.767408

Table 7
Statistical properties of natural frequency for differents boundary conditions (1089 samples)

Statistics	Boundary conditions			
	Simply supported	Clamped–free	Clamped–clamped	Clamped supported
Mean	672.4186337	672.0966348	674.762564	673.1879431
Standard deviation	0.000767083	0.002587103	0.0046755	0.006558834
Coefficient of variation	1.14078E−06	3.8493E−06	6.9291E−06	9.74295E−06

(4) The intricate connection of the subject of this study with the inverse problems should be mentioned. As Gladwell (1996) stresses, "classical direct problems have involved the analysis and derivation of the *behavior* of the system (e.g. forced response, natural frequencies, current flow, stresses, etc.) from its *properties* such as density or mass, conductivity, elastic constant, crack lengths, etc. Inverse problem are concerned with the determination or estimation of such properties from behavior." It turns out that although each of the beams has different boundary condition, and, moreover, each of them has different $E(\xi)$ these beams have the same frequency. Two vibrating system which have the same natural frequencies are called to be *isospectral*. In our particular case, beams of different boundary conditions share the *first* natural frequency. Gottlieb (1991), Driscoll (1997) and others have constructed examples of isospectral structures. In particular, Gottlieb (1991) (consult also with Gottlieb, 1992 and Gottlieb and Mc Manus, 1998) showed that clamped *inhomogeneous* circular plates have the same vibration spectrum as their homogeneous counterparts. In our cases, the second and other frequencies do not coincide. For example, the second natural frequency squared of the simply supported beam is $10881.18b\, I/(AL^4)$, while the clamped–clamped beam has a second natural frequency squared $5607.68\ I/(AL^4)$. Clamped–free and clamped-supported beams second natural frequencies squared are respectively $42727.97\ I/(AL^4)$ and $8013.24\ I/(AL^4)$. These values are obtained by the finite element method. Difference between the present work and those associated with the inverse vibration problem lies in our desire of obtaining closed-form solutions to find *any* beam that has a polynomial mode shape.

(5) The expressions of the squared fundamental frequency depend solely upon two coefficients a_m and b_{m+4}. If by any procedure these coefficients may be made at *will* of the designer, one can have a beam that has a *pre-selected* fundamental natural frequency so that the unwanted resonance condition can be avoided. Whereas we are unaware of a procedure with such a derivable feature at present, its possible development in the future cannot be a priori ruled out.

(6) If the coefficients a_m and b_{m+4} are deterministic but the remaining coefficients are random, the natural fundamental frequency squared is a deterministic quantity. In complex structures that must be analyzed by approximate methods this remarkable phenomenon could be validated if the coefficient of variation of the output quantity turns out to be much smaller than its counterparts for the input parameters.

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Appendix A. Beam clamped at both ends ($m \leqslant 3$)

A.1. Clamped–clamped beam with uniform material density ($m = 0$)

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0, \qquad E(\xi) = \sum_{i=0}^{4} b_i \xi^i. \tag{A.1}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{3} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{4} i(i-1)b_i\xi^i + 12\sum_{i=0}^{2}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{4} b_i\xi^i \\ &\quad -48\sum_{i=0}^{3}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{4} ib_i\xi^i - kL^4a_0\left(\xi^2 - 2\xi^3 + \xi^4\right) = 0. \end{aligned} \tag{A.2}$$

Eq. (A.2) has to be satisfied for any ξ. This requirement yields

$$24(b_0 - b_1) + 4b_2 = 0, \tag{A.3}$$

$$72(b_1 - b_2) + 12b_3 = 0, \tag{A.4}$$

$$144(b_2 - b_3) + 24b_4 - kL^4a_0 = 0, \tag{A.5}$$

$$240b_3 - 240b_4 + 2L^4ka_0 = 0, \tag{A.6}$$

$$-kL^4a_0 + 360b_4 = 0. \tag{A.7}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3\}$, has to be

$$b_3 = -2b_4, \tag{A.8}$$

$$b_2 = \frac{b_4}{3}, \tag{A.9}$$

$$b_1 = \frac{2b_4}{3}, \tag{A.10}$$

$$b_0 = \frac{11b_4}{18}. \tag{A.11}$$

To sum up, if conditions (A.1) are satisfied, where b_i are given by Eqs. (A.8)–(A.11), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 360\frac{I}{A}\frac{b_4}{a_0L^4}. \tag{A.12}$$

It is remarkable that this formula coincides with that for the clamped–clamped beam. A word of caution is in order: although the expressions of ω_1^2 coalesce, the rest of b_j coefficients, namely b_0, b_1, b_2, and b_3 given in Eqs. (23)–(26) and in Eqs. (A.8)–(A.11) differ.

A.2. Clamped–clamped beam with linearly varying material density $(m = 1)$

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0 + a_1\xi, \qquad E(\xi) = \sum_{i=0}^{5} b_i \xi^i. \tag{A.13}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{4} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{5} i(i-1)b_i\xi^i + 12\sum_{i=0}^{3}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{5} b_i\xi^i \\ &\quad -48\sum_{i=0}^{4}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{5} i b_i\xi^i - kL^4(a_0 + a_1\xi)\left(\xi^2 - 2\xi^3 + \xi^4\right) = 0. \end{aligned} \tag{A.14}$$

Eq. (A.14) has to be satisfied for any ξ. This requirement yields

$$24(b_0 - b_1) + 4b_2 = 0, \tag{A.15}$$

$$72(b_1 - b_2) + 12b_3 = 0, \tag{A.16}$$

$$144(b_2 - b_3) + 24b_4 - kL^4 a_0 = 0, \tag{A.17}$$

$$240(b_3 - b_4) + 40b_5 + L^4(2ka_0 - ka_1) = 0, \tag{A.18}$$

$$360(b_4 - b_5) + L^4(2ka_1 - ka_0) = 0, \tag{A.19}$$

$$-kL^4 a_1 + 504b_5 = 0. \tag{A.20}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3, 4\}$, has to be

$$b_4 = \frac{7a_0 - 9a_1}{5a_1} b_5, \tag{A.21}$$

$$b_3 = \frac{2(a_1 - 21a_0)}{15a_1} b_5, \tag{A.22}$$

$$b_2 = \frac{13a_1 + 14a_0}{30a_1} b_5, \tag{A.23}$$

$$b_1 = \frac{37a_1 + 84a_0}{90a_1} b_5, \tag{A.24}$$

$$b_0 = \frac{61a_1 + 154a_0}{180a_1} b_5. \tag{A.25}$$

To sum up, if conditions (A.13) are satisfied, where b_i are given by Eqs. (A.21)–(A.25), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 504 \frac{I}{A} \frac{b_5}{a_1 L^4}. \tag{A.26}$$

A.3. Clamped–clamped beam with parabolically varying density (m = 2)

In this case, the expressions of $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0 + a_1\xi + a_2\xi^2, \qquad E(\xi) = \sum_{i=0}^{6} b_i\xi^i. \tag{A.27}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{5} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{6} i(i-1)b_i\xi^i + 12\sum_{i=0}^{4}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{6} b_i\xi^i \\ &\quad -48\sum_{i=0}^{5}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{6} ib_i\xi^i - kL^4\left(a_0 + a_1\xi + a_2\xi^2\right)\left(\xi^2 - 2\xi^3 + \xi^4\right) = 0. \end{aligned} \tag{A.28}$$

Eq. (A.28) has to be satisfied for any ξ. This requirement yields

$$24(b_0 - b_1) + 4b_2 = 0, \tag{A.29}$$

$$72(b_1 - b_2) + 12b_3 = 0, \tag{A.30}$$

$$144(b_2 - b_3) + 24b_4 - kL^4a_0 = 0, \tag{A.31}$$

$$240(b_3 - b_4) + 40b_5 + L^4(2ka_0 - ka_1) = 0, \tag{A.32}$$

$$360(b_4 - b_5) + 60b_6 + L^4(2ka_1 - ka_0 - ka_2) = 0, \tag{A.33}$$

$$504(b_5 - b_6) + L^4(2ka_2 - ka_1) = 0, \tag{A.34}$$

$$-kL^4a_2 + 672b_6 = 0. \tag{A.35}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3, 4, 5\}$, has to be

$$b_5 = \frac{4a_1 - 5a_2}{3a_2} b_6, \tag{A.36}$$

$$b_4 = \frac{a_2 - 72a_1 + 56a_0}{30a_2} b_6, \tag{A.37}$$

$$b_3 = \frac{2(7a_2 + 4a_1 - 84a_0)}{45a_2} b_6, \tag{A.38}$$

$$b_2 = \frac{55a_2 + 104a_1 + 112a_0}{180a_2} b_6, \tag{A.39}$$

$$b_1 = \frac{137a_2 + 296a_1 + 672a_0}{540a_2} b_6, \tag{A.40}$$

$$b_0 = \frac{219a_2 + 488a_1 + 1232a_0}{1080a_2} b_6. \tag{A.41}$$

To sum up, if conditions (A.27) are satisfied, where b_i are given by Eqs. (A.36)–(A.41), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 672\frac{I}{A}\frac{b_6}{a_2L^4}. \tag{A.42}$$

A.4. Clamped–clamped beam with cubic density ($m = 3$)

In this case, the expressions of $E(\xi)$ and $\rho(\xi)$ read

$$E(\xi) = \sum_{i=0}^{7} b_i\xi^i, \qquad \rho(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3. \tag{A.43}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{6} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{7} i(i-1)b_i\xi^i + 12\sum_{i=0}^{5}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{7} b_i\xi^i \\ &\quad -48\sum_{i=0}^{6}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{7} ib_i\xi^i - kL^4\left(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3\right)\left(\xi^2 - 2\xi^3 + \xi^4\right) = 0. \end{aligned} \tag{A.44}$$

Eq. (A.44) has to be satisfied for any ξ. This requirement yields

$$24(b_0 - b_1) + 4b_2 = 0, \tag{A.45}$$

$$72(b_1 - b_2) + 12b_3 = 0, \tag{A.46}$$

$$144(b_2 - b_3) + 24b_4 - kL^4a_0 = 0, \tag{A.47}$$

$$240(b_3 - b_4) + 40b_5 + L^4(2ka_0 - ka_1) = 0, \tag{A.48}$$

$$360(b_4 - b_5) + 60b_6 + L^4(2ka_1 - ka_0 - ka_2) = 0, \tag{A.49}$$

$$504(b_5 - b_6) + 84b_7 + L^4(2ka_2 - ka_1 - ka_3) = 0, \tag{A.50}$$

$$672(b_6 - b_7) + L^4(2ka_3 - ka_2) = 0, \tag{A.51}$$

$$-kL^4a_3 + 864b_7 = 0. \tag{A.52}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3, 4, 5, 6\}$, has to be

$$b_6 = \frac{9a_2 - 11a_3}{7a_3}b_7, \tag{A.53}$$

$$b_5 = \frac{72a_1 - 90a_2 - a_3}{42a_3}b_7, \tag{A.54}$$

$$b_4 = \frac{50a_3 + 9a_2 - 648a_1 + 504a_0}{210a_3}b_7, \tag{A.55}$$

$$b_3 = \frac{305a_3 + 504a_2 + 288a_1 - 6048a_0}{1260a_3}b_7, \tag{A.56}$$

$$b_2 = \frac{85a_3 + 165a_2 + 312a_1 + 336a_0}{420a_3} b_7, \tag{A.57}$$

$$b_1 = \frac{1225a_3 + 2466a_2 + 5328a_1 + 12096a_0}{7560a_3} b_7, \tag{A.58}$$

$$b_0 = \frac{970a_3 + 1971a_2 + 4392a_1 + 11088a_0}{7560a_3} b_7. \tag{A.59}$$

To sum up, if conditions (A.43) are satisfied, where b_i are given by Eqs. (A.53)–(A.59), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 864 \frac{I}{A} \frac{b_7}{a_3 L^4}. \tag{A.60}$$

Appendix B. Beam clamped at one end and simply supported at the other ($m \geqslant 3$) (CS)

B.1. CS beam with uniform density ($m = 0$)

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0, \qquad E(\xi) = \sum_{i=0}^{4} b_i \xi^i. \tag{B.1}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{3} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{4} i(i-1)b_i\xi^i + 12\sum_{i=0}^{2}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{4} b_i\xi^i \\ &\quad -48\sum_{i=0}^{3}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{4} ib_i\xi^i - kL^4a_0\left(2\xi^2 - 5\xi^3 + 3\xi^4\right) = 0. \end{aligned} \tag{B.2}$$

Eq. (B.2) has to be satisfied for any ξ. This requirement yields

$$48b_0 - 60b_1 + 12b_2 = 0, \tag{B.3}$$

$$144b_1 - 180b_2 + 36b_3 = 0, \tag{B.4}$$

$$288b_2 - 360b_3 + 72b_4 - 3kL^4a_0 = 0, \tag{B.5}$$

$$480b_3 - 600b_4 + L^4(5ka_0 - 3ka_1) = 0, \tag{B.6}$$

$$-kL^4a_0 + 360b_4 = 0. \tag{B.7}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3\}$, has to be

$$b_3 = -\frac{5b_4}{2}, \tag{B.8}$$

$$b_2 = \frac{3b_4}{8}, \tag{B.9}$$

$$b_1 = \frac{35b_4}{32}, \tag{B.10}$$

$$b_0 = \frac{163b_4}{128}. \tag{B.11}$$

To sum up, if conditions (B.1) are satisfied, where b_i are given by Eqs. (B.8)–(B.11), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 360\frac{I}{A}\frac{b_4}{a_0L^4}. \tag{B.12}$$

We again the identity of this formula with it counterparts for clamped–clamped beam in Eq. (20) and for the cantilever beam in Eq. (A.12). Again, this coincidence occurs if both beams share the same coefficients b_4 and a_0. Yet we see from the very construction process of the closed-form solutions that the beams are not identical in rest of b_j coefficients.

B.2. CS beam with linearly varying density ($m = 1$)

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0 + a_1\xi, \qquad E(\xi) = \sum_{i=0}^{5} b_i\xi^i. \tag{B.13}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{4} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{5} i(i-1)b_i\xi^i + 12\sum_{i=0}^{3}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{5} b_i\xi^i \\ &\quad - 48\sum_{i=0}^{4}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{5} ib_i\xi^i - kL^4(a_0 + a_1\xi)\left(3\xi^2 - 5\xi^3 + 2\xi^4\right) = 0. \end{aligned} \tag{B.14}$$

Eq. (B.14) has to be satisfied for any ξ. This requirement yields

$$48b_0 - 60b_1 + 12b_2 = 0, \tag{B.15}$$

$$144b_1 - 180b_2 + 36b_3 = 0, \tag{B.16}$$

$$288b_2 - 360b_3 + 72b_4 - 3kL^4a_0 = 0, \tag{B.17}$$

$$480b_3 - 600b_4 + 120b_5 + L^4(5ka_0 - 3ka_1) = 0, \tag{B.18}$$

$$720b_4 - 900b_5 + L^4(5ka_1 - 2ka_0) = 0, \tag{B.19}$$

$$-kL^4a_1 + 504b_5 = 0. \tag{B.20}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3, 4\}$, has to be

$$b_4 = \frac{28a_0 - 45a_1}{20a_1}b_5, \tag{B.21}$$

$$b_3 = \frac{7(a_1 - 40a_0)}{80a_1} b_5, \tag{B.22}$$

$$b_2 = \frac{215a_1 + 168a_0}{320a_1} b_5, \tag{B.23}$$

$$b_1 = \frac{1047a_1 + 1960a_0}{1280a_1} b_5, \tag{B.24}$$

$$b_0 = \frac{7(625a_1 + 1304a_0)}{5120a_1} b_5. \tag{B.25}$$

To sum up, if conditions (B.13) are satisfied, where b_i are given by Eqs. (B.21)–(B.25), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 504 \frac{I}{A} \frac{b_5}{a_1 L^4}. \tag{B.26}$$

B.3. CS beam with parabolically varying density $(m = 2)$

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$E(\xi) = \sum_{i=0}^{6} b_i \xi^i, \qquad \rho(\xi) = a_0 + a_1 \xi + a_2 \xi^2. \tag{B.27}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{5} i(i+1) b_{i+1} \xi^i + 12\sum_{i=2}^{6} i(i-1) b_i \xi^i + 12\sum_{i=0}^{4} (i+1)(i+2) b_{i+2} \xi^i + 24\sum_{i=0}^{6} b_i \xi^i \\ &\quad - 48\sum_{i=0}^{5} (i+1) b_{i+1} \xi^i + 48\sum_{i=1}^{6} i b_i \xi^i - kL^4 \left(a_0 + a_1 \xi + a_2 \xi^2\right)\left(3\xi^2 - 5\xi^3 + 2\xi^4\right) = 0. \end{aligned} \tag{B.28}$$

Eq. (B.28) has to be satisfied for any ξ. This requirement yields

$$48b_0 - 60b_1 + 12b_2 = 0, \tag{B.29}$$

$$144b_1 - 180b_2 + 36b_3 = 0, \tag{B.30}$$

$$288b_2 - 360b_3 + 72b_4 - 3kL^4 a_0 = 0, \tag{B.31}$$

$$480b_3 - 600b_4 + 120b_5 + L^4(5ka_0 - 3ka_1) = 0, \tag{B.32}$$

$$720b_4 - 900b_5 + 180b_6 + L^4(5ka_1 - 2ka_0 - 3ka_2) = 0, \tag{B.33}$$

$$1008b_5 - 1260b_6 + L^4(5ka_2 - 2ka_1) = 0, \tag{B.34}$$

$$-kL^4 a_2 + 672b_6 = 0. \tag{B.35}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3, 4, 5\}$, has to be

$$b_5 = \frac{16a_1 - 25a_2}{12a_2} b_6, \tag{B.36}$$

$$b_4 \frac{-13a_2 - 720a_1 + 448a_0}{240a_2} b_6, \tag{B.37}$$

$$b_3 = \frac{435a_2 + 112a_1 - 4480a_0}{960a_2} b_6, \tag{B.38}$$

$$b_2 = \frac{2227a_2 + 3440a_1 + 2688a_0}{3840a_2} b_6, \tag{B.39}$$

$$b_1 = \frac{9395a_2 + 16752a_1 + 31360a_0}{15360a_2} b_6, \tag{B.40}$$

$$b_0 = \frac{38067a_2 + 70000a_1 + 146048a_0}{61440a_2} b_6. \tag{B.41}$$

To sum up, if conditions (B.27) are satisfied, where b_i are given by Eqs. (B.36)–(B.41), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 672 \frac{I}{A} \frac{b_6}{a_2 L^4}. \tag{B.42}$$

B.4. CS beam with cubically varying density $(m = 3)$

In this case, the expressions for $E(\xi)$ and $\rho(\xi)$ read

$$E(\xi) = \sum_{i=0}^{7} b_i \xi^i, \qquad \rho(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3. \tag{B.43}$$

By substituting the latter expressions in Eq. (3), we obtain

$$\begin{aligned} &-24\sum_{i=1}^{6} i(i+1)b_{i+1}\xi^i + 12\sum_{i=2}^{7} i(i-1)b_i\xi^i + 12\sum_{i=0}^{5}(i+1)(i+2)b_{i+2}\xi^i + 24\sum_{i=0}^{7} b_i\xi^i \\ &\quad - 48\sum_{i=0}^{6}(i+1)b_{i+1}\xi^i + 48\sum_{i=1}^{7} ib_i\xi^i - kL^4\left(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3\right)\left(3\xi^2 - 5\xi^3 + 2\xi^4\right) = 0. \end{aligned} \tag{B.44}$$

Eq. (B.44) has to be satisfied for any ξ. This requirement yields

$$48b_0 - 60b_1 + 12b_2 = 0, \tag{B.45}$$

$$144b_1 - 180b_2 + 36b_3 = 0, \tag{B.46}$$

$$288b_2 - 360b_3 + 72b_4 - 3kL^4 a_0 = 0, \tag{B.47}$$

$$480b_3 - 600b_4 + 120b_5 + L^4(5ka_0 - 3ka_1) = 0, \tag{B.48}$$

$$720b_4 - 900b_5 + 180b_6 + L^4(5ka_1 - 2ka_0 - 3ka_2) = 0, \tag{B.49}$$

$$1008b_5 - 1260b_6 + 252b_7 + L^4(5ka_2 - 2ka_1 - 3ka_3) = 0, \tag{B.50}$$

$$1344b_6 - 1680b_7 + L^4(5ka_3 - 2ka_2) = 0, \tag{B.51}$$

$$-kL^4a_3 + 864b_7 = 0. \tag{B.52}$$

To satisfy the compatibility equations, b_i, where $i = \{0, 1, 2, 3, 4, 5, 6\}$, has to be

$$b_6 = \frac{36a_2 - 55a_3}{28a_3}b_7, \tag{B.53}$$

$$b_5 = \frac{3(64a_1 - 100a_2 - 5a_3)}{112a_3}b_7, \tag{B.54}$$

$$b_4 = \frac{725a_3 - 156a_2 - 8640a_1 + 5376a_0}{2240a_3}b_7, \tag{B.55}$$

$$b_3 = \frac{3925a_3 + 5220a_2 + 1344a_1 - 53760a_0}{8960a_3}b_7, \tag{B.56}$$

$$b_2 = \frac{3(5575a_3 + 8908a_2 + 13760a_1 + 10752a_0)}{35840a_3}b_7, \tag{B.57}$$

$$b_1 = \frac{67925a_3 + 112740a_2 + 201024a_1 + 376320a_0}{143360a_3}b_7, \tag{B.58}$$

$$b_0 = \frac{272725a_3 + 456804a_2 + 840000a_1 + 1752576a_0}{573440a_3}b_7. \tag{B.59}$$

To sum up, if conditions (B.43) are satisfied, where b_i are given by Eqs. (B.53)–(B.59), then the fundamental mode shape is expressed by Eq. (6), where the fundamental natural frequency reads

$$\omega^2 = 864\frac{I}{A}\frac{b_7}{a_3L^4}. \tag{B.60}$$

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